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Cohomogeneity-one G_2-structures

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Abstract

G_2-manifolds with a cohomogeneity-one action of a compact Lie group G are studied. For G simple, all solutions with holonomy G_2 and weak holonomy G_2 are classified. The holonomy G_2 solutions are necessarily Ricci-flat and there is a one-parameter family with SU(3)-symmetry. The weak holonomy G_2 solutions are Einstein of positive scalar curvature and are uniquely determined by the simple symmetry group. During the proof the equations for G_2-symplectic and G_2-cosymplectic structures are studied and the topological types of the manifolds admitting such structures are determined. New examples of compact G_2-cosymplectic manifolds and complete G_2-symplectic structures are found.

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1. Introduction

A G_2-structure on a seven-dimensional manifold M is an identification of the tangent space with the imaginary octonians. Equivalently, the geometry is determined by a three-form ϕ which at each point is of 'generic type', in that it lies in a particular open orbit for the action of $\mathrm{GL}(7,\mathbb{R})$ (such forms are 'stable' in Hitchin's terminology [14]). The three-form ϕ determines a Riemannian metric g and hence a Hodge-star operator $*$.

If ϕ and the four-form $*\phi$ are both closed, then g is Ricci-flat and has holonomy contained in G_2. This is one of the two exceptional holonomy groups in the Berger classification (see

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[2,6]). The first non-trivial complete examples were constructed by Bryant and Salamon [7] and compact examples have since been found by Joyce (first in [15,16] and more recently in [17]) and by Kovalev [19].

If $d\phi = \lambda * \phi$, for some non-zero constant λ, then g is an Einstein metric of positive scalar curvature and M is said to have weak holonomy G_2. This terminology was first introduced by Gray [13]. Many homogeneous examples are known. For example, each Aloff–Walach space $SU(3)/U(1)_{k,\ell}$, where $U(1)_{k,\ell} = \{\mathrm{diag}(e^{ik\theta}, e^{i\ell\theta}, e^{-i(k+\ell)\theta})\}$ and k, ℓ are integers, carries two such metrics (see [8]). As k and ℓ vary, this family includes infinitely many different homeomorphisms types. A classification of the compact homogeneous manifolds with weak holonomy G_2 is given in [11].

In this paper, we study G_2-structures with a cohomogeneity-one action of a compact Lie group G. This means that G acts on M preserving the three-form ϕ and that the generic orbit on M has dimension $7 - 1 = 6$. We will first determine the connected groups G that can act. Thereafter, we study the equations for holonomy and weak holonomy G_2-structures in the case that G is simple and determine all solutions. The simple groups in question are G_2, Sp(2) and SU(3). In each case, we find that the weak holonomy G_2 solutions are unique; they are only complete in the case with symmetry G_2, and here one gets the round metric on the seven-sphere (and its quotient $\mathbb{R}P(7)$). The limited number of solutions is in strong contrast to the homogeneous case. For holonomy G_2, the solutions for the first two symmetry groups are isolated, whereas for SU(3) there is a one-parameter family of solutions. This family contains a unique complete metric, which turns out to have U(3)-symmetry. The G_2-symmetric solution is flat, whereas those with symmetry Sp(2) and U(3) are the metrics found by Bryant and Salamon [7]. In private communications, Andrew Dancer and McKenzie Wang, and Gary Gibbons and Chris Pope tell us that they have also recently found the one-dimensional family of triaxial SU(3)-symmetric metrics. Note that by considering non-simple symmetry groups new complete metrics with holonomy G_2 have been found by Brandhuber et al. [3].

Both weak holonomy and holonomy structures satisfy $d * \phi = 0$ and so are special examples of cosymplectic G_2-structures. Any hypersurface in an eight-manifold with holonomy Spin(7) carries a cosymplectic G_2-structure and homogeneous cosymplectic G_2-structures with symmetry Sp(2) are behind the new Spin(7)-holonomy examples constructed in [10]. Our approach gives examples of compact cohomogeneity-one manifolds with cosymplectic G_2-structures. By Hitchin [14] these are hypersurfaces in manifolds of holonomy Spin(7). It is therefore an interesting question for future work, which of these Spin(7) metrics are complete.

The other part of the holonomy G_2-equations is $d\phi = 0$. Solutions to this equation define what are known as symplectic G_2-structures. We show that for cohomogeneity-one manifolds with simple symmetry group, a symplectic G_2-structure exist only if the manifold also admits a holonomy G_2 metric.

2. G_2-structures

Let W be $\mathbb{R}^7$ with its usual inner product g_0. Take $\{v^0, \dots, v^6\}$ to be an orthonormal basis for W and write $v_{01} = v_0 v_1 = v_0 \wedge v_1$, etc., in the exterior algebra $\Lambda^* W^*$. For each

$\theta \in \mathbb{R}$, we define a three-form $\phi(\theta)$ on W by

$$\phi(\theta) = \omega_0 \wedge v_0 + \cos\theta\,\alpha_0 + \sin\theta\,\beta_0, \tag{2.1}$$

where $\alpha_0 = v_{246} - v_{235} - v_{145} - v_{136}$, $\beta_0 = v_{135} - v_{146} - v_{236} - v_{245}$ and $\omega_0 = v_{12} + v_{34} + v_{56}$.

The Lie group G_2 may be defined to be the stabiliser of $\phi(0)$ under the action of $\mathrm{GL}(7, \mathbb{R})$. From this, Bryant shows that G_2 is a compact, connected, simply-connected Lie group of dimension 14 [5]. The subgroup of G_2 fixing v^0 is isomorphic to SU(3). Indeed, in the basis $u_0 = v_0$, $u_k = v_{2k-1} + \mathrm{i}v_{2k}$, $k = 1, 2, 3$, for $W^* \otimes \mathbb{C}$, we have

$$\phi(\theta) = \tfrac{1}{2}\mathrm{i}((u_1\bar{u}_1 + u_2\bar{u}_2 + u_3\bar{u}_3)u_0 + \mathrm{e}^{-\mathrm{i}\theta}u_1u_2u_3 - \mathrm{e}^{\mathrm{i}\theta}\bar{u}_1\bar{u}_2\bar{u}_3).$$

Thus, $\phi(\theta) = \mathrm{e}^{-\mathrm{i}\theta/3}\phi(0)$ showing that stabilisers of $\phi(\theta)$ are all conjugate in SO(7) and that $6g_0(v, w)\,\mathrm{vol}_0 = (v \lrcorner \phi(\theta)) \wedge (w \lrcorner \phi(\theta)) \wedge \phi(\theta)$ is independent of θ.

Conversely, the Lie group G_2 acts transitively on the unit sphere in $\mathbb{R}^7$. A choice of unit vector v^0, determines a stabiliser isomorphic to SU(3) and the action of SU(3) on $\langle v^0\rangle^\perp$ fixes a Kähler form ω_0 and a complex volume which may be written as $\mathrm{e}^{\mathrm{i}\theta}u_1u_2u_3$. In this way, we see that there is an orthonormal basis so that the G_2 three-form is $\phi(\theta)$ as in (2.1).

A G_2-structure on a seven-dimensional manifold M is specified by fixing a three-form ϕ such that for each p there is a basis of $W = T_pM$ so that $\phi_p = \phi(\theta)$ for some θ. We say that a compact Lie group G acts on (M^7, ϕ) with cohomogeneity-one if G preserves the three-form ϕ and the largest G-orbits are of dimension 6. In this case, $B = M/G$ is a one-dimensional manifold, quite possibly with boundary. The orbits lying over the interior of B are all isomorphic to G/K, where $K = K_p$ is the stabiliser of a $p \in M$ with $G \cdot p \in \mathrm{Int}\,B$. We call these orbits *principal* and any remaining orbits are called *special*. Let G/H be a special orbit. Using the action of G, we may assume that H is a subgroup of K. A necessary and sufficient condition for M to be a smooth manifold is that for each special orbit G/H, the quotient H/K is a sphere [20].

3. Principal orbit structure

The requirement that G acts on M with cohomogeneity-one preserving ϕ implies that the representation of the isotropy group $K = K_p$ on the tangent space of a principal orbit is as a subgroup of SU(3) on its standard six-dimensional representation $[\![\Lambda^{1,0}]\!] \cong \mathbb{R}^6$. Considering the Lie algebras only we find that $\mathfrak{k}$ must be isomorphic to either $\mathfrak{su}(3)$, $\mathfrak{u}(2)$, $\mathfrak{su}(2)$, $2\mathfrak{u}(1)$, $\mathfrak{u}(1)$ or $\{0\}$. The possible isotropy representations are then the real representations underlying the following three-dimensional complex representations: the standard representation of $\mathfrak{su}(3)$, the representation $L^2 \oplus \bar{L}V$ of $\mathfrak{u}(2)$, the representations S^2V and $\mathbb{C} \oplus V$ of $\mathfrak{su}(2)$, the representation $L_1 \oplus L_2 \oplus \bar{L}_1\bar{L}_2$ of $2\mathfrak{u}(1)$, the representation $L \oplus \bar{L} \oplus \mathbb{C}$ of $\mathfrak{u}(1)$ and finally the trivial representation $3\mathbb{C}$ of $\{0\}$. For each of the non-trivial representations U of a possible isotropy algebra $\mathfrak{k}$ the direct sum $\mathfrak{g} = \mathfrak{k} \oplus U$ happens to determine a unique compact real Lie algebra. These are, respectively, $\mathfrak{g}_2$, $\mathfrak{sp}(2)$, $3\mathfrak{su}(2)$, $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$, $\mathfrak{su}(3)$ and $2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The trivial representation may be taken to represent either $2\mathfrak{su}(2)$, $\mathfrak{su}(2) \oplus 3\mathfrak{u}(1)$ or $6\mathfrak{u}(1)$.

If, on the other hand, G/K is any effective six-dimensional homogeneous space with K acting on the isotropy representation as a subgroup of SU(3), then we may pick an invariant Kähler form ω and an invariant complex volume form α on G/K and obtain a non-degenerate three-form on $M = \mathbb{R} \times G/K$ by defining $\phi = \mathrm{d}t \wedge \omega + \mathrm{Re}(\alpha)$.

Theorem 3.1. *Let (M^7, ϕ) be a G_2-manifold of cohomogeneity-one under a compact, connected Lie group. Then, as almost effective homogeneous spaces, the principal orbits are one of the following*:

$$S^6 = \frac{G_2}{\mathrm{SU}(3)}, \qquad \mathbb{C}P(3) = \frac{\mathrm{Sp}(2)}{\mathrm{SU}(2)\mathrm{U}(1)}, \qquad F_{1,2} = \frac{\mathrm{SU}(3)}{T^2},$$
$$S^3 \times S^3 = \frac{\mathrm{SU}(2)^3}{\mathrm{SU}(2)} = \frac{\mathrm{SU}(2)^2 T^1}{T^1} = \mathrm{SU}(2)^2,$$
$$S^5 \times S^1 = \frac{\mathrm{SU}(3)T^1}{\mathrm{SU}(2)}, \qquad S^3 \times (S^1)^3 = \mathrm{SU}(2)T^3, \qquad (S^1)^6 = T^6,$$

up to finite quotients. Conversely, any cohomogeneity-one manifold with one of these as principal orbit carries a G_2-structure.

In this paper, we will consider the case when G is simple. The principal orbits are the first three cases listed above. The first of these is distinguished from the other two in that K acts irreducibly on U.

4. Irreducible isotropy

This is the case when the principal orbit is $G_2/\mathrm{SU}(3)$. The isotropy representation is the real module underlying the standard representation $\Lambda^{1,0} \cong \mathbb{C}^3$ of SU(3). Up to scale this admits precisely one invariant two-form ω and one invariant symmetric two-tensor g_0. The space of invariant three-forms is two-dimensional, spanned by α and β. We fix the scales as follows. Set g_0 to be the canonical metric on $S^6 = G_2/\mathrm{SU}(3)$ with sectional curvature one. Then let ω, α and β be such that $\omega^3 = 6\,\mathrm{vol}_0$, $\mathrm{d}\omega = 3\alpha$, $*_0\alpha = \beta$ and $\mathrm{d}\beta = -2\omega^2$.

Let γ be a geodesic through p orthogonal to the principal orbit $G_2/\mathrm{SU}(3)$ and parameterise γ by arc-length $t \in I \subset \mathbb{R}$. Then the union of principal orbits is $I \times G_2/\mathrm{SU}(3) \subset M$ and there are smooth functions $f, \theta : I \to \mathbb{R}$ such that

$$g = \mathrm{d}t^2 + f^2 g_0, \qquad \mathrm{vol} = f^6\,\mathrm{vol}_0 \wedge \mathrm{d}t, \tag{4.1}$$

$$\phi = f^2 \omega \wedge \mathrm{d}t + f^3(\cos\theta\,\alpha + \sin\theta\,\beta). \tag{4.2}$$

Note that $f(t)$ is non-zero for each $t \in I$. Our choice of scales now gives

$$*\phi = \tfrac{1}{2} f^4\omega^2 + f^3(\cos\theta\,\beta - \sin\theta\,\alpha) \wedge \mathrm{d}t, \qquad \mathrm{d}{*\phi} = 2f^3(f' - \cos\theta)\omega^2 \wedge \mathrm{d}t,$$
$$\mathrm{d}\phi = (3f^2 - (f^3\cos\theta)')\alpha \wedge \mathrm{d}t - (f^3\sin\theta)'\beta \wedge \mathrm{d}t - 2f^3\sin\theta\,\omega^2.$$

We first consider the cosymplectic G_2-equations $\mathrm{d}{*\phi} = 0$ which are equivalent to $f' = \cos\theta$. Locally, these are described by the one arbitrary function θ. Alternatively, one may regard them as determined by solutions to the differential inequality $|f'| \le 1$.

Geometrically the solutions may be understood as follows. Consider $\mathbb{R}^8 = W \times \mathbb{R}$ with its standard Spin(7) four-form $\Omega = \phi(0) \wedge v_8 + *_7\phi(0)$. As $\mathrm{Spin}(7) = \mathrm{stab}_{\mathrm{GL}(8,\mathbb{R})}\,\Omega$ acts transitively on the unit sphere in $\mathbb{R}^8$ with stabiliser G_2, we see that for any unit vector N, the three-form $N \lrcorner \Omega$ defines a G_2-structure on $\langle N\rangle^\perp$ and that $\Omega = \phi \wedge N^\flat + *\phi$. As Ω is closed we, therefore, have Gray's observation that any oriented hypersurface $H \subset \mathbb{R}^8$ with unit normal N carries a cosymplectic G_2-structure.

The hypersurface $H = \{(v, s) \in W \times \mathbb{R} : \|v\| = r(s)\}$ is of cohomogeneity-one under the action of G_2. Its metric is $(1 + (\mathrm{d}r/\mathrm{d}s)^2)\,\mathrm{d}s^2 + r^2 g_0$. Reparameterising so that $\mathrm{d}t = \sqrt{(1 + (\mathrm{d}r/\mathrm{d}s)^2)}\,\mathrm{d}s$, we obtain a metric in the form (4.1) with $f(t) = r(s(t))$ and hence $f'(t) = (\mathrm{d}r/\mathrm{d}s)/\sqrt{1 + (\mathrm{d}r/\mathrm{d}s)^2}$. However, this has $|f'(t)| < 1$, so we may write $f' = \cos\theta$ and we see that locally each cosymplectic G_2-solution is given this way away from $|\cos\theta| = 1$.

The symplectic G_2-equations $\mathrm{d}\phi = 0$ imply first that $\sin\theta \equiv 0$. We then get $|\cos\theta| = 1$ and $f' = \cos\theta$, so such metrics are also cosymplectic and have holonomy G_2. However, the solutions are simply $f(t) = \pm t$ and we get the standard flat metric on $\mathbb{R}^7$ with its standard G_2-structure.

The equations $\mathrm{d}\phi = \lambda *\phi$ for weak holonomy G_2 give

$$\lambda f = -4\sin\theta \quad \text{and} \quad 4\theta' = -\lambda.$$

Thus, $f(t) = (4/\lambda)\sin(\lambda t/4)$. The hypersurface discussion above shows that this is locally the round metric on S^7.

5. Reducible isotropy: the equations

Let us begin with the case of SU(3)-symmetry. The principal isotropy group $K = T^2 = S^1_1 \times S^1_2$ acts on the standard representation $\Lambda^{1,0} \cong \mathbb{C}^3$ as $L_1 + L_2 + \bar{L}_1\bar{L}_2$, where $L_i \cong \mathbb{C}$, are the standard representations of $S^1_i \cong \mathrm{U}(1)$. Using the isomorphism $\mathfrak{su}(3) \otimes \mathbb{C} \cong \Lambda^{1,1}_0$, we find that the isotropy representation is $[\![L_1\bar{L}_2]\!] + [\![L_1L_2^2]\!] + [\![L_1^2L_2]\!]$. Each irreducible submodule carries an invariant metric g_i and symplectic form ω_i, $i = 1, 2, 3$, but the space of invariant three-forms has dimension 2. Identifying T^2 with the diagonal matrices in SU(3), we fix the basis

$$E_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$E_4 = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_5 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_6 = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of the tangent space at the origin and let $\{e_1, \dots, e_6\}$ denote the dual basis. We may now write

$$g_1 = e_1^2 + e_2^2, \qquad g_2 = e_3^2 + e_4^2, \qquad g_3 = e_5^2 + e_6^2,$$
$$\omega_1 = e_{12}, \qquad \omega_2 = e_{34}, \qquad \omega_3 = e_{56},$$

and find that

$$\alpha = e_{246} - e_{235} - e_{145} - e_{136}, \qquad \beta = e_{135} - e_{146} - e_{236} - e_{245}$$

is a basis for the invariant three-forms. Put $\mathrm{vol}_0 = e_{123456}$. As left-invariant one-forms on $SU(3)$ we have $\mathrm{d}e_i(E_j, E_k) = e_i([E_j, E_k])$. One may thus show that on $SU(3)/T^2$ one has

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = \mathrm{d}\omega_3 = \tfrac{1}{2}\alpha, \quad \mathrm{d}\alpha = 0, \qquad \mathrm{d}\beta = -2(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1) \quad \text{and}$$
$$\mathrm{d}(\omega_i\omega_j) = 0. \tag{5.1}$$

Any $SU(3)$-invariant G_2-structure on $I \times SU(3)/T^2$ has

$$g = \mathrm{d}t^2 + f_1^2 g_1 + f_2^2 g_2 + f_3^2 g_3, \qquad \mathrm{vol} = f_1^2 f_2^2 f_3^2\, \mathrm{vol}_0 \wedge \mathrm{d}t, \tag{5.2}$$

where $t \in I \subset \mathbb{R}$ is the arc-length parameter of an orthogonal geodesic and f_i are non-vanishing functions. Using the equation $(X \lrcorner \phi) \wedge (Y \lrcorner \phi) \wedge \phi = 6g(X, Y)\,\mathrm{vol}$ and normalisation $\phi \wedge *\phi = 7\,\mathrm{vol}$, we find that the corresponding invariant three-form is

$$\phi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge \mathrm{d}t + f_1 f_2 f_3(\cos\theta\, \alpha + \sin\theta\, \beta) \tag{5.3}$$

for some function $\theta(t)$. The G_2-structure now has

$$*\phi = f_2^2 f_3^2\omega_2\omega_3 + f_3^2 f_1^2\omega_3\omega_1 + f_1^2 f_2^2\omega_1\omega_2 + f_1 f_2 f_3(\cos\theta\, \beta - \sin\theta\, \alpha) \wedge \mathrm{d}t,$$

and hence

$$\begin{aligned}
\mathrm{d} * \phi = {} & ((f_2^2 f_3^2)' - 2 f_1 f_2 f_3 \cos\theta)\omega_2\omega_3 \wedge \mathrm{d}t \\
& + ((f_3^2 f_1^2)' - 2 f_1 f_2 f_3 \cos\theta)\omega_3\omega_1 \wedge \mathrm{d}t \\
& + ((f_1^2 f_2^2)' - 2 f_1 f_2 f_3 \cos\theta)\omega_1\omega_2 \wedge \mathrm{d}t, \\
\mathrm{d}\phi = {} & (\tfrac{1}{2}(f_1^2 + f_2^2 + f_3^2) - (f_1 f_2 f_3 \cos\theta)')\alpha \wedge \mathrm{d}t \\
& - (f_1 f_2 f_3 \sin\theta)'\beta \wedge \mathrm{d}t - 2 f_1 f_2 f_3 \sin\theta(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1).
\end{aligned}$$

We therefore have that the $SU(3)$-invariant G_2-structure is cosymplectic if

$$(f_1^2 f_2^2)' = (f_3^2 f_1^2)' = (f_2^2 f_3^2)' = 2 f_1 f_2 f_3 \cos\theta. \tag{5.4}$$

It is G_2-symplectic if

$$(f_1 f_2 f_3 \cos\theta)' = \tfrac{1}{2}(f_1^2 + f_2^2 + f_3^2) \quad \text{and} \quad f_1 f_2 f_3 \sin\theta = 0. \tag{5.5}$$

The equations for weak holonomy G_2 are

$$(f_1 f_2 f_3 \cos\theta)' = \tfrac{1}{2}(f_1^2 + f_2^2 + f_3^2) + \lambda f_1 f_2 f_3 \sin\theta, \tag{5.6a}$$

$$(f_1 f_2 f_3 \sin\theta)' = -\lambda f_1 f_2 f_3 \cos\theta, \tag{5.6b}$$

$$-2 f_1 f_2 f_3 \sin\theta = \lambda f_1^2 f_2^2 = \lambda f_2^2 f_3^2 = \lambda f_3^2 f_1^2. \tag{5.6c}$$

Let us now consider the case of $Sp(2)$-symmetry. The principal isotropy group $K = U(1) \times Sp(1)$ acts on the standard representation $E \cong \mathbb{C}^4$ as $E \cong H + L + \bar{L}$, where $H \cong \mathbb{C}^2$ and $L \cong \mathbb{C}$ are the standard representations of $Sp(1) = SU(2)$ and $U(1)$, respectively.

Using $\mathfrak{sp}(2)\otimes\mathbb{C}\cong S^2E$ we find that the isotropy representation is $[\![L^2]\!]+[\![H\bar{L}]\!]$. Both of these modules carry an invariant metric g_i and symplectic form ω_i. The space of invariant three-forms on their sum is two-dimensional. We give the isotropy representation the basis

$$E_1=\frac{1}{2}\begin{pmatrix}0&0\\0&\mathrm{j}\end{pmatrix},\qquad E_2=\frac{1}{2}\begin{pmatrix}0&0\\0&-\mathrm{k}\end{pmatrix},\qquad E_3=\frac{1}{2\sqrt{2}}\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$
$$E_4=\frac{1}{2\sqrt{2}}\begin{pmatrix}0&\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\qquad E_5=\frac{1}{2\sqrt{2}}\begin{pmatrix}0&\mathrm{j}\\\mathrm{j}&0\end{pmatrix},\qquad E_6=\frac{1}{2\sqrt{2}}\begin{pmatrix}0&\mathrm{k}\\\mathrm{k}&0\end{pmatrix}.$$

Then the dual elements $\{e_1,\dots,e_6\}$ are such that $\{e_1,e_2\}$ is a basis for $[\![L^2]\!]^*$ and $\{e_3,\dots,e_6\}$ is a basis for $[\![H\bar{L}]\!]^*$. We scale g_i and ω_i so that

$$g_1=e_1^2+e_2^2,\qquad g_2=e_3^2+e_4^2+e_5^2+e_6^2,\qquad \omega_1=e_{12},\qquad \omega_2=e_{34}+e_{56}.$$

Then

$$\alpha=e_{246}-e_{235}-e_{145}-e_{136},\qquad \beta=e_{135}-e_{146}-e_{236}-e_{245}$$

is a basis for the invariant three-forms. Put $\mathrm{vol}_0=e_{123456}$. Using the Lie algebra structure of $\mathfrak{sp}(2)$, one finds that the corresponding left-invariant forms on $\mathrm{Sp}(2)/(\mathrm{U}(1)\times\mathrm{Sp}(1))$ satisfy

$$\mathrm{d}\omega_1=\tfrac{1}{2}\alpha,\qquad \mathrm{d}\omega_2=\alpha,\qquad \mathrm{d}\alpha=0\quad\text{and}\quad \mathrm{d}\beta=-2\omega_1\omega_2-\omega_2^2.$$

Proceeding as in the SU(3)-case one finds that the Sp(2)-invariant G_2-structures are given by Eqs. (5.2) and (5.3) with $f_3\equiv f_2$. Computing further, one finds that the equations for these structures to be cosymplectic, symplectic or have weak holonomy G_2 are those for SU(3)-symmetry with $f_3\equiv f_2$. We may therefore treat Sp(2)-symmetry as if it were a special case of SU(3)-symmetry.

6. Solving the cosymplectic G_2-equations

Consider the cosymplectic G_2-equation (5.4). The differences of the differentials gives that $f_i^2(f_j^2-f_k^2)$ is constant for any permutation (ijk) of (123). We may therefore relable the f_i so that $f_3^2\geq f_2^2\geq f_1^2\geq 0$ for all t and write

$$f_1^2(f_3^2-f_2^2)=\mu^2,\qquad f_2^2(f_3^2-f_1^2)=\nu^2,\qquad f_3^2(f_2^2-f_1^2)=\nu^2-\mu^2\tag{6.1}$$

for some constants $\nu\geq\mu\geq 0$.

Let us first deal with two special cases. If $\nu=0$, then $f_1^2=f_2^2=f_3^2$ and we are left with the equation

$$2f_1'=\pm\cos\theta.$$

Up to a factor of 2 this is just the equation obtained for G_2-symmetry in Section 4. Note that we have $|f_1'|\leq 1/2$.

If $\nu>\mu=0$, then $2f_2^2=f_1^2+\sqrt{f_1^4+4\nu^2}$ and $f_1'=\cos\theta\left(1+f_1^2/\sqrt{f_1^4+4\nu^2}\right)^{-1}$, with θ an arbitrary function. Note that in this case $|f_1'|\leq 1$ and $|f_2'|=|f_1\cos\theta/2f_2|<1/2$.

The general case is $\nu \geq \mu > 0$. Here $f_3^2 > f_2^2 \geq f_1^2 > 0$ and Eq. (6.1) may be rearranged to give

$$f_2^2 + \nu^2 f_2^{-2} = f_3^2 + (\nu^2 - \mu^2) f_3^{-2}, \tag{6.2a}$$

$$f_3^2 - (\nu^2 - \mu^2) f_3^{-2} = f_1^2 + \mu^2 f_1^{-2}, \tag{6.2b}$$

$$f_1^2 - \mu^2 f_1^{-2} = f_2^2 - \nu^2 f_2^{-2}. \tag{6.2c}$$

Regarding Eqs. (6.2a)–(6.2c) as quadratic in f_i^2, one sees that the corresponding discriminants are non-negative.

Let $\Delta(i; j)$ be the discriminant of (6.2a) with respect to f_j^2. Then we have

$$\begin{aligned}
\Delta_1 &:= \Delta(2;3) = (f_1^2 + \mu^2 f_1^{-2})^2 + 4(\nu^2 - \mu^2) \\
&= (f_1^2 - \mu^2 f_1^{-2})^2 + 4\nu^2 = \Delta(3;2) = (f_3^2 + (\nu^2 - \mu^2) f_3^{-2})(f_2^2 + \nu^2 f_2^{-2}), \\
\Delta_2 &:= \Delta(3;1) = (f_2^2 - \nu^2 f_2^{-2})^2 + 4\mu^2 = (f_2^2 + \nu^2 f_2^{-2})^2 - 4(\nu^2 - \mu^2) \\
&= \Delta(1;3) = (f_1^2 + \mu^2 f_1^{-2})(f_3^2 - (\nu^2 - \mu^2) f_3^{-2}), \\
\Delta_3 &:= \Delta(1;2) = (f_3^2 + (\nu^2 - \mu^2) f_3^{-2})^2 - 4\nu^2 \\
&= (f_3^2 - (\nu^2 - \mu^2) f_3^{-2})^2 - 4\mu^2 = \Delta(2;1) = (f_2^2 - \nu^2 f_2^{-2})(f_1^2 - \mu^2 f_1^{-2}).
\end{aligned}$$

The positivity of Δ_3 written as $\Delta(1; 2)$ implies that $f_3^4 - 2\nu f_3^2 + \nu^2 \geq \mu^2$ which in turns gives either $f_3^2 \leq \nu - \mu$ or $f_3^2 \geq \nu + \mu$. However, Eq. (6.2b) implies that $f_3^4 > \nu^2 - \mu^2 = (\nu + \mu)(\nu - \mu) > (\nu - \mu)^2$, so

$$f_3^2 \geq \nu + \mu.$$

Also Eq. (6.2c) implies that $\varepsilon = \operatorname{sgn}(f_1^2 - \mu) = \operatorname{sgn}(f_2^2 - \nu)$ is well defined. Using these remarks, we can choose consistent branches of square roots in solving the quadratic equations (6.2a)–(6.2c). For example, solving (6.2c) for f_2^2 and writing the discriminant as a function of f_1^2, we get

$$\begin{aligned}
(f_1^2 f_2^2)' &= \frac{1}{2}(f_1^4 + f_1^2\sqrt{\Delta_1} - \mu^2)' = 2(f_1^4 + f_1^2\sqrt{\Delta_1} - \mu^2 + 2\nu^2)\frac{f_1^3 f_1'}{\sqrt{\Delta_1}} \\
&= 4(f_1^2 f_2^2 + \nu^2)\frac{f_1^3 f_1'}{\sqrt{\Delta_1}} = \frac{4 f_2^2 f_3^2 f_1^3 f_1'}{\sqrt{\Delta_1}}.
\end{aligned}$$

Doing similar computations for the other $(f_i^2 f_j^2)'$ and putting the results into (5.4) gives

$$f_1' = \tfrac{1}{2} f_2^{-1} f_3^{-1} \cos\theta \sqrt{\Delta_1} = \tfrac{1}{2}\varepsilon_{23} \cos\theta \sqrt{(1 + \nu^2 f_2^{-4})(1 + (\nu^2 - \mu^2) f_3^{-4})}, \tag{6.3a}$$

$$f_2' = \tfrac{1}{2} f_3^{-1} f_1^{-1} \cos\theta \sqrt{\Delta_2} = \tfrac{1}{2}\varepsilon_{31} \cos\theta \sqrt{(1 - (\nu^2 - \mu^2) f_3^{-4})(1 + \mu^2 f_1^{-4})}, \tag{6.3b}$$

$$f_3' = \tfrac{1}{2}\varepsilon f_1^{-1} f_2^{-1} \cos\theta \sqrt{\Delta_3} = \tfrac{1}{2}\varepsilon_{12}^* \cos\theta \sqrt{(1 - \mu^2 f_1^{-4})(1 - \nu^2 f_2^{-4})}, \tag{6.3c}$$

where $\varepsilon_{ij} = \operatorname{sgn}(f_i f_j)$ and $\varepsilon_{ij}^* = \varepsilon_{ij}\varepsilon$. We may rewrite the right-hand side of Eq. (6.3a) so that it only contains θ and f_1. Then for a given function θ, we get an implicit differential equation for f_1:

$$f_1' = \varepsilon \cos\theta \sqrt{\Xi(f_1, \mu, \nu)}, \tag{6.4}$$

where

$$\Xi(f_1, \mu, \nu) = \frac{f_1^8 + 2(2\nu^2 - \mu^2) f_1^4 + \mu^4}{2 f_1^4 \left(f_1^4 + (2\nu^2 - \mu^2) + \sqrt{f_1^8 + 2(2\nu^2 - \mu^2) f_1^4 + \mu^4} \right)}. \tag{6.5}$$

Note that this function $\Xi(f_1, \mu, \nu)$ is positive and decreasing with

$$\lim_{|f_1| \to \infty} \Xi(f_1, \mu, \nu) = \tfrac{1}{4}.$$

Alternatively, the structure may be determined by the function f_1.

Theorem 6.1. *Consider a cosymplectic G_2-structure preserved by an action of* SU(3) *of cohomogeneity-one. Then the metric is given by Eq.* (5.2)*. Arrange the coefficients so that* $f_3^2 \geq f_2^2 \geq f_1^2$*. Then*

$$|f_1'| \leq \sqrt{\Xi(f_1, \mu, \nu)} \tag{6.6}$$

for some constants $\nu \geq \mu \geq 0$.

Conversely, any smooth function f_1 satisfying the differential inequality (6.6) *gives a cosymplectic G_2-structure with f_2 determined by Eq.* (6.2c), *f_3 by Eq.* (6.2b) *and θ by* $f_3 \cos\theta = (f_1 f_2)'$.

Note that by rescaling and reparameterising we may rid ourselves of one of the parameters and, for example, when $\mu \neq 0$ set either μ, ν or $\mu + \nu$ equal to 1.

The case of Sp(2)-symmetry is now obtained by setting either $\mu = 0$ or $\mu = \nu$.

Theorem 6.2. *Consider a cosymplectic G_2-structure preserved by an action of* Sp(2) *of cohomogeneity-one. Then the metric is given by* (5.2) *with $f_3 = f_2$. The difference $f_1^2 - f_2^2$ has constant sign. If $f_1^2 \leq f_2^2$, then*

$$2 f_2^2 = f_1^2 + \sqrt{f_1^4 + 4\nu^2} \quad \textit{and} \quad |f_1'| \leq \frac{\sqrt{f_1^4 + 4\nu^2}}{f_1^2 + \sqrt{f_1^4 + 4\nu^2}}$$

for some $\nu \geq 0$. If $f_1^2 \geq f_2^2$, then

$$2 f_1^2 = f_2^2 + \nu^2 f_2^{-2} \quad \textit{and} \quad |f_2'| \leq \frac{\sqrt{f_2^4 + 4\nu^2}}{2 f_2^2}$$

for some $\nu \geq 0$.

Conversely, any smooth functions f_1 and f_2 satisfying the above equations determine a cosymplectic G_2-structure.

Again, we may rescale and reparameterise to obtain $\nu = 0$ or 1.

7. Topology and boundary conditions

Let us now turn to discussion of the possible topologies of manifolds with G_2-structure and a compact simple symmetry group G acting with cohomogeneity-one. General references for the cohomogeneity-one situation may be found in [1,4].

Let M be a manifold of cohomogeneity-one under G with principal isotropy group K and base $B = M/G$. The possible topologies for B are homeomorphic to either $\mathbb{R}$, S^1, $[0, \infty)$ or $[0, 1]$. In the first case, M is homeomorphic to the product $\mathbb{R} \times G/K$ and an invariant tensor τ on M is smooth if and only if τ is smooth considered as a function from $\mathbb{R}$ to the space of K-invariant tensors on the isotropy representation of the principal orbit.

When the base is a circle, the total space M is homeomorphic to a quotient

$$\mathbb{R} \times_h \frac{G}{K},$$

where (t, gK) is identified with $(t + 1, ghK)$ for some element $h \in N_G(K)$, the normaliser of K in G. Given h and h' in $N_G(K)$, these determine the same manifold if $hK = h'K$ and they determine equivariantly diffeomorphic manifolds if they satisfy $fhf^{-1} = h'$ for some $f \in N_G(K)$. For the principal orbits in question this translates into periodicity requirements corresponding to the different orders of the elements of $N_G(K)/K$. An invariant tensor τ_t must satisfy

$$h^*\tau_t = \tau_{t+1}$$

to be well defined.

When the base is a half-open interval, the end point is the image of a special orbit with isotropy group H, where H/K is diffeomorphic to a sphere $S^m \subset V \simeq \mathbb{R}^{m+1}$ for some representation V of H. The total space M is then diffeomorphic to the vector bundle

$$M \cong G \times_H V \to \frac{G}{H}.$$

We note that if $x \in S^m$ has isotropy K and $h \in H$ satisfies $h \cdot x = -x$ then h defines an element $hK \in N_G(K)/K$ of order 2. Conversely, any non-trivial element hK of $N_G(K)/K$ of order 2 defines a subgroup $H \subset G$ with H/K a sphere by taking $H = K \cup hK$. An invariant tensor τ_t on M must now satisfy

$$h^*\tau_t = \tau_{-t}$$

if it is smooth. This requirement is in general only sufficient when $H/K \cong \mathbb{Z}_2$. If H/K has positive dimension, a metric two-tensor on $M_0 = M \setminus \pi^{-1}(\{0\})$ extends to a smooth metric on M under the following two conditions. Firstly, the induced metric $g_t(H/K)$ on

$(0, \infty) \times H/K \subset M_0$ should satisfy

$$g_t\left(\frac{H}{K}\right) = dt^2 + f^2(t)g_0,$$

where g_0 is the standard metric on the sphere with sectional curvature one and f is an odd function with $|f'(0)| = 1$. Secondly, $g_t(X, X)$ should be positive everywhere for Killing vector fields induced by elements of $\mathfrak{h}^\perp \subset \mathfrak{g}$. For the cases we consider, a G_2-structure on M_0 defined by a three-from ϕ extends to a smooth G_2-structure on M if and only if $h^*\phi_t = \phi_{-t}$ and the metric defined by ϕ extends to a smooth metric on M, see Section 10.

Finally, consider the situation where B is a closed interval. Let $\pi\colon M \to B$ be the projection. Then the subspaces $M_0 = \pi^{-1}[0, 1)$ and $M_1 = \pi^{-1}(0, 1]$ are diffeomorphic to vector bundles $G \times_{H_i} V_i \to G/H_i$, where H_i acts transitively on the unit sphere in V_i with isotropy K. Given G, K, H_0 and H_1, the possible diffeomorphism types of M with principal isotropy group K and special isotropy groups H_0 and H_1 are parameterised by the double coset space $N_0\backslash N_G(K)/N_1$, where $N_i := N_G(K) \cap N_G(H_i)$. These double cosets correspond to the different equivariant identifications we may make of $M_0 \setminus \pi^{-1}\{0\}$ with $M_1 \setminus \pi^{-1}\{1\}$. The boundary conditions on tensors in this case are obtained from those for the case of one singular orbit by considering their restrictions to the half-open intervals.

We will employ the following notation for spaces of cohomogeneity-one with special orbits. When the base M/G is homeomorphic to the half-open interval we write $M = [G/H|G/K)$, where G/H is the special orbit over the end point and G/K the principal orbit. When the base is a closed interval we write $M = [G/H_0|G/K|G/H_1]$.

We now turn to more detailed consideration of our particular principal orbit types.

8. Solutions: irreducible isotropy

This is the case of symmetry G_2 with principal orbit $G_2/\mathrm{SU}(3)$. The normaliser of $\mathrm{SU}(3)$ is

$$N_{G_2}(\mathrm{SU}(3)) = \mathrm{SU}(3) \bigcup D_7\,\mathrm{SU}(3),$$

where $D_7 = \mathrm{diag}(-1, 1, -1, 1, -1, 1, -1)$. To each of these two elements of $N_{G_2}(\mathrm{SU}(3))/\mathrm{SU}(3)$ corresponds a quotient $\mathbb{R} \times_h G_2/\mathrm{SU}(3)$ with base diffeomorphic to a circle.

There are precisely two special orbit types: $\mathbb{R}P(6) = G_2/N_{G_2}(\mathrm{SU}(3))$ and a point $\{*\} = G_2/G_2$. To these correspond firstly two spaces with base homeomorphic to $[0, \infty)$. The first is the canonical line bundle over $\mathbb{R}P(6)$, the second is $\mathbb{R}^7$ viewed as a seven-dimensional vector bundle over a point.

There are three spaces with $B = [0, 1]$ corresponding to the three possible choices of two special orbits. If both special orbits are points the space in question is S^7; when one is a point and the other is $\mathbb{R}P(6)$ the space is diffeomorphic to $\mathbb{R}P(7)$; and when both are $\mathbb{R}P(6)$ we obtain the connected sum $\mathbb{R}P(7)\#\mathbb{R}P(7)$. The corresponding double coset spaces have precisely one element and therefore there is only one diffeomorphism type in

Table 1
G_2 solutions with symmetry G_2

M^7	Holonomy and symplectic	Weak holonomy	Cosymplectic
S^7	None	Complete	Complete
$\mathbb{R}P(7)$	None	Complete	Complete
$\mathbb{R}P(7)\#\mathbb{R}P(7)$	None	None	Complete
$S^1 \times S^6$	None	None	Complete
$\mathbb{R} \times_{D_7} S^6$	*No G_2-structure*		
$[\mathbb{R}P(6)\|S^6)$	None	None	Complete
$\mathbb{R}^7$	Complete	Incomplete	Complete
$\mathbb{R} \times S^6$	Incomplete	Incomplete	Complete

each case. The action of D_7 on the invariant tensors of S^6 is

$$D_7^*(g_0, \omega, \alpha, \beta) = (g_0, -\omega, -\alpha, \beta).$$

As a consequence $D_7^* \operatorname{vol}_0 = -\operatorname{vol}_0$. In particular, the space $\mathbb{R} \times_{D_7} S^6$ is not orientable and therefore cannot carry a G_2-structure.

When M has a special orbit with isotropy G_2 at $t = 0$ the metric g_t extends to a smooth metric on a neighbourhood of the special orbit if and only if the function f is odd with $|f'(0)| = 1$. The requirement $D_7^*\phi_t = \phi_{-t}$ now implies that $\sin\theta$ is odd and $\cos\theta$ is even around $t = 0$. If, on the other hand, the special orbit at $t = 0$ is $\mathbb{R}P(6)$, then f must be even and non-zero everywhere for the metric to extend smoothly. In that case $\cos\theta$ must be even and $\sin\theta$ odd.

Now consider the cosymplectic equations. One solution is given by $f \equiv c$, where c is a positive constant, and $\theta \equiv 0$. This solution satisfies the boundary conditions for $[\mathbb{R}P(6)|S^6|\mathbb{R}P(6)]$ and $[\mathbb{R}P(6)|S^6)$, as well as the periodicity requirement for $\mathbb{R} \times_e S^6 = S^1 \times S^6$.

The unique solution to the symplectic equation satisfies the boundary conditions only for $\mathbb{R}^7 = [*|S^6)$. The weak holonomy solutions $f(t) = 4\lambda^{-1}\sin(\lambda t/4)$ are smooth on $S^7 = [*|S^6|*]$ for $t \in [0, 4\pi/\lambda]$ and on $\mathbb{R}P(7) = [*|S^6|\mathbb{R}P(6)]$ for $t \in [0, 2\pi/\lambda]$. Different choices of λ scale the metric by a homothety.

Theorem 8.1. *Let M^7 be a manifold with G_2-structure preserved by an action of G_2 of cohomogeneity-one. Then the principal orbit is $G_2/\mathrm{SU}(3)$ and M^7 is listed in* Table 1. *The symplectic G_2, holonomy G_2 and weak holonomy G_2 solutions are unique up to scale. The first two are flat, the last has constant curvature.*

9. Solutions: reducible isotropy

We first consider the instance of SU(3)-symmetry; that for Sp(2) will then follow relatively easily. The principal orbits are $\mathrm{SU}(3)/T^2$ and the normaliser of T^2 in SU(3) is

$$N_{\mathrm{SU}(3)}(T^2) = \bigcup_{\sigma \in \Sigma_3} A_\sigma T^2,$$

where Σ_3 is the symmetric group on three elements and

$$A_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_{(23)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore there are three spaces $\mathbb{R} \times_{A_\sigma} F_{1,2}$ over the circle corresponding to $A_{(23)}$, $A_{(123)}$ and e.

There are two special orbit types: $\mathbb{C}P(2)_1 = \mathrm{SU}(3)/\mathrm{U}(2)_{(23)}$ and $\mathcal{F}_{(23)} = F_{1,2}/A_{(23)} = \mathrm{SU}(3)/(T^2 \cup A_{(23)}T^2)$, where $\mathrm{U}(2)_{(23)}$ is the $\mathrm{U}(2) \subset \mathrm{SU}(3)$ containing $T^2 \cup A_{(23)}T^2$. Corresponding to these there are two spaces with base homeomorphic to the half-open interval.

The double coset spaces $N_0 \backslash N_{\mathrm{SU}(3)}(T^2)/N_1$ all have two components. Therefore, we have six different cohomogeneity-one spaces with the closed interval as base. When both special orbits are complex projective spaces, we may write the space from the trivial double coset as $[\mathbb{C}P(2)_1|F_{1,2}|\mathbb{C}P(2)_1]$ and that from the non-trivial double coset as $[\mathbb{C}P(2)_1|F_{1,2}|\mathbb{C}P(2)_2]$, where $\mathbb{C}P(2)_2 = \mathrm{SU}(3)/\mathrm{U}(2)_{(13)}$. We use similar notation in the other cases.

Now consider in turn the actions of the elements $A_{(23)}$ and $A_{(123)}$. The element $A_{(23)}$ acts with order 2 and transforms the $\mathrm{SU}(3)$-invariant tensors of $F_{1,2}$ as

$$A^*_{(23)}(g_1, g_2, g_3, \omega_1, \omega_2, \omega_3, \alpha, \beta) = (g_1, g_3, g_2, -\omega_1, -\omega_3, -\omega_2, -\alpha, \beta).$$

This implies that the manifold $\mathbb{R} \times_{A_{(23)}} F_{1,2}$ cannot carry a G_2-structure. It also leads to boundary conditions on the metric and three-form for the two types of special orbit. For special orbit $\mathbb{C}P(2)_1$ these translate into

f_1 and $\sin\theta$ are odd functions,

$\cos\theta$ is an even function,

$f_2^2(t) = f_3^2(-t)$, $|f_1'(0)| = 1$ and $f_2(0) \neq 0$.

Those for the special orbit $\mathcal{F}_{(23)}$ are

f_1 and $\sin\theta$ are even functions,

$\cos\theta$ is an odd function,

$f_2^2(t) = f_3^2(-t)$ and $f_1(0) \neq 0 \neq f_2(0)$.

Note that in both cases the product $f_2 f_3$ is even.

The action of $A_{(123)}$ on the invariant tensors of $F_{1,2}$ is

$$A^*_{(123)}(g_1, g_2, g_3, \omega_1, \omega_2, \omega_3, \alpha, \beta) = (g_2, g_3, g_1, \omega_2, \omega_3, \omega_1, \alpha, \beta),$$

whence the periodicity conditions on $\mathbb{R} \times_{A_{(123)}} F_{1,2}$ state that $f_1^2(t) = f_2^2(t+1) = f_3^2(t+2)$. Note that the tensors

$$g_0 = g_1 + g_2 + g_3, \qquad \omega_0 = \omega_1 + \omega_2 + \omega_3, \qquad \alpha \quad \text{and} \quad \beta$$

all are invariant under

$$T^{(123)} = \bigcup_{\sigma\ \text{even}} A_\sigma T^2. \tag{9.1}$$

Therefore, $\mathcal{F}_{(123)} = \mathrm{SU}(3)/T^{(123)}$ is a second possible principal orbit for symmetry SU(3). It is not hard to check that $A_{(123)}$ generates the only possible finite action on the principal orbits that preserves an SU(3)-invariant G_2-structure. The normaliser of $T^{(123)}$ in SU(3) is of course $T^{(123)} \cup A_{(23)}T^{(123)}$ and $A_{(23)}$ acts on the invariant tensors as

$$A^*_{(23)}(g_0, \omega_0, \alpha, \beta) = (g_0, -\omega_0, -\alpha, \beta).$$

For the principal orbit $\mathcal{F}_{(123)}$, the analysis is now the same as for the case of G_2-symmetry discussed in the previous section.

Returning to principal orbit $F_{1,2}$, we see that taking $f_1 = f_2 = f_3 \equiv c$ with c a positive constant, and $\theta \equiv 0$ solves the SU(3)-symmetric cosymplectic equations as well as the periodicity requirement on $S^1 \times F_{1,2}$ and $\mathbb{R} \times_{A_{(123)}} F_{1,2}$ and the boundary conditions on $[\mathcal{F}_{(23)}|F_{1,2})$, $[\mathcal{F}_{(23)}|F_{1,2}|\mathcal{F}_{(23)}]$ and $[\mathcal{F}_{(23)}|F_{1,2}|\mathcal{F}_{(13)}]$.

Consider the cosymplectic G_2-equations together with the boundary conditions for either $[\mathbb{C}P(2)_1|F_{1,2}|\mathbb{C}P(2)_2]$ or $[\mathbb{C}P(2)_1|F_{1,2}|\mathcal{F}_{(13)}]$. From (6.1), we have that two of the three constants μ, ν and $\nu^2 - \mu^2$ must be zero. But this implies that the third constant is also zero and that $f_1^2 = f_2^2 = f_3^2$. The boundary conditions now give that f_1 is both even and odd at $t = 0$ which clearly cannot be the case. Thus these spaces do not carry invariant cosymplectic G_2-structures.

Finally, let us consider the cosymplectic equations on $[\mathbb{C}P(2)_1|F_{1,2}|\mathbb{C}P(2)_1]$ and $[\mathbb{C}P(2)_1|F_{1,2}|\mathcal{F}_{(23)}]$. Solutions on these spaces can be obtained as follows. Set

$$\mathrm{d}\theta = (1 + \sin^2\theta)^{1/4}\,\mathrm{d}t,$$

and determine the remaining functions via Eq. (6.4). The metric is then

$$g = (1 + \sin^2\theta)^{-1/2}(\mathrm{d}\theta^2 + \sin^2\theta g_1 + (1 + \sin^2\theta)(g_2 + g_3)),$$

and the three-form is

$$\begin{aligned}\phi &= (1 + \sin^2\theta)^{-3/4}((\sin^2\theta\,\omega_1 + (1 + \sin^2\theta)(\omega_2 + \omega_3))\,\mathrm{d}\theta \\ &\quad + \sin\theta(1 + \sin^2\theta)(\cos\theta\,\alpha + \sin\theta\,\beta)).\end{aligned}$$

With $\theta \in [0, \pi]$ these solve the cosymplectic equations and the boundary conditions for $[\mathbb{C}P(2)_1|F_{1,2}|\mathbb{C}P(2)_1]$. Restricting θ to $[0, \pi/2]$ we also get a solution on $[\mathbb{C}P(2)_1|F_{1,2}|\mathcal{F}_{(23)}]$.

This completes the discussion of the cosymplectic equations under SU(3)-symmetry. We will return to the holonomy and weak holonomy equations after discussing the symmetry group Sp(2).

Theorem 9.1. *Let M^7 be a manifold with G_2-structure preserved by an action of* SU(3) *of cohomogeneity-one. The principal orbit is either $F_{1,2} = \mathrm{SU}(3)/T^2$ or its $\mathbb{Z}_3$-quotient $\mathcal{F}_{(123)} = \mathrm{SU}(3)/T^{(123)}$, see (9.1). The possible M^7 are listed in Table 2 together with information on the existence of cosymplectic G_2-structures.*

The topological analysis in the case of Sp(2)-symmetry is very similar to the G_2 case for the simple reason that the normaliser of U(1) Sp(1) again has two components:

Table 2
G_2 solutions with symmetry SU(3). Here $\mathcal{F}_\sigma = F_{1,2}/A_\sigma$ and $\mathcal{F}_\Sigma = F_{1,2}/\Sigma_3$

M^7	Holonomy and symplectic	Weak holonomy	Cosymplectic
$[\mathbb{C}P(2)_1 \| F_{1,2} \| \mathbb{C}P(2)_1]$	None	None	Complete
$[\mathbb{C}P(2)_1 \| F_{1,2} \| \mathbb{C}P(2)_2]$	None	None	None
$[\mathbb{C}P(2)_1 \| F_{1,2} \| \mathcal{F}_{(23)}]$	None	None	Complete
$[\mathbb{C}P(2)_1 \| F_{1,2} \| \mathcal{F}_{(13)}]$	None	None	None
$[\mathcal{F}_{(23)} \| F_{1,2} \| \mathcal{F}_{(23)}]$	None	None	Complete
$[\mathcal{F}_{(23)} \| F_{1,2} \| \mathcal{F}_{(13)}]$	None	None	Complete
$S^1 \times F_{1,2}$	None	None	Complete
$\mathbb{R} \times_{A_{(23)}} F_{1,2}$	*No G_2-structure*		
$\mathbb{R} \times_{A_{(123)}} F_{1,2}$	None	None	Complete
$[\mathbb{C}P(2)_1 \| F_{1,2})$	Complete	None	Complete
$[\mathcal{F}_{(23)} \| F_{1,2})$	None	None	Complete
$\mathbb{R} \times F_{1,2}$	Incomplete	Incomplete	Complete
$[\mathcal{F}_\Sigma \| \mathcal{F}_{(123)} \| \mathcal{F}_\Sigma]$	None	None	Complete
$[\mathcal{F}_\Sigma \| \mathcal{F}_{(123)})$	None	None	Complete
$S^1 \times \mathcal{F}_{(123)}$	None	None	Complete
$\mathbb{R} \times_{A_{(23)}} \mathcal{F}_{(123)}$	No G_2-structure		
$\mathbb{R} \times \mathcal{F}_{(123)}$	Incomplete	Incomplete	Complete

$$N_{\mathrm{Sp}(2)}(\mathrm{U}(1)\,\mathrm{Sp}(1)) = \mathrm{U}(1)\,\mathrm{Sp}(1) \bigcup D_2\,\mathrm{U}(1)\,\mathrm{Sp}(1),$$

where

$$D_2 = \begin{pmatrix} \mathrm{j} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we have two possible spaces $\mathbb{R} \times_h \mathbb{C}P(3)$ with base a circle, and two possible special orbit types:

$$S^4 = \mathbb{H}P(1) = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}, \qquad \mathcal{C} = \frac{\mathbb{C}P(3)}{\mathbb{Z}_2} = \frac{\mathrm{Sp}(2)}{N_{\mathrm{Sp}(2)}(\mathrm{U}(1)\,\mathrm{Sp}(1))}.$$

Let us now consider the action of D_2 on the invariant tensors of $\mathbb{C}P(3)$:

$$D_2^*(g_1, g_2, \omega_1, \omega_2, \alpha, \beta) = (g_1, g_2, -\omega_1, -\omega_2, \alpha, -\beta).$$

This means that the boundary conditions are precisely those for SU(3)-symmetry with $f_2 \equiv f_3$. In particular, the compact, complete solutions to the cosymplectic equations found for SU(3)-symmetry also give solutions for Sp(2)-symmetry. The results of the analysis in this case may be found in Table 3. Note that the existence of an invariant G_2-structure implies that the only possible principal orbit is $\mathbb{C}P(3)$.

Let us now turn to the weak holonomy equations, firstly for SU(3). Eqs. (5.6c) and (6.1) imply that $f_1^2 = f_2^2 = f_3^2$ and that $f_i = -\varepsilon_{jk} 2\lambda^{-1} \sin\theta$. Eqs. (5.6a) and (5.6b) are now

$$(\theta' + 4\lambda)\sin^2\theta(4\cos^2\theta - 1) = 0, \qquad (\theta' + 4\lambda)\sin^3\theta\cos\theta = 0.$$

As f_i is non-zero on the principal orbits, we get that $\theta' = -4\lambda$. We deduce that we have the same behaviour for Sp(2)-symmetry.

Table 3
G_2 solutions with symmetry Sp(2). Here $\mathcal{C}$ denotes $\mathbb{C}P(3)/\mathbb{Z}_2$

M^7	Holonomy and symplectic	Weak holonomy	Cosymplectic
$[S^4\vert\mathbb{C}P(3)\vert S^4]$	None	None	Complete
$[S^4\vert\mathbb{C}P(3)\vert\mathcal{C}]$	None	None	Complete
$[\mathcal{C}\vert\mathbb{C}P(3)\vert\mathcal{C}]$	None	None	Complete
$S^1 \times \mathbb{C}P(3)$	None	None	Complete
$\mathbb{R} \times_{D_2} \mathbb{C}P(3)$	No G_2-structure		
$[S^4\vert\mathbb{C}P(3))$	Complete	None	Complete
$[\mathcal{C}\vert\mathbb{C}P(3))$	None	None	Complete
$\mathbb{R} \times \mathbb{C}P(3)$	Incomplete	Incomplete	Complete

Theorem 9.2. *Up to scale, the spaces* $(0, \pi/2) \times G/K$, *with* $G/K = F_{1,2}, \mathcal{F}_{(123)}$ *or* $\mathbb{C}P(2)$ *admit unique structures with weak holonomy* G_2 *invariant under the action of* SU(3) *or* Sp(2). *The metric and three-forms are*

$$g = 4\,\mathrm{d}\theta^2 + \sin^2\theta\, g_0, \qquad \phi = \sin^2\theta(\omega_0 \wedge \mathrm{d}\theta + \sin\theta(\cos\theta\,\alpha + \sin\theta\,\beta)),$$

where $g_0 = \sum_i g_i$ *and* $\omega_0 = \sum_i \omega_i$. *These structures are incomplete and do not extend over any special orbits.*

Next we discuss the holonomy solutions. The second equation in (5.5) implies that $\sin\theta \equiv 0$. Let $\varepsilon_\theta = \cos\theta$. Then the cosymplectic equations (5.4) show that the first equation in (5.5) is automatically satisfied. We thus have that f_1 satisfies the differential equation

$$f_1' = \varepsilon_\theta \sqrt{\Xi(f_1, \mu, \nu)},$$

where Ξ is defined in (6.5). As $\Xi(f_1, \mu, \nu) \geq 1/4$, we have that $|f_1| \geq (1/2)t + c$ and so any complete solution has exactly one special orbit and f_1 vanishes on that orbit.

If $\nu = 0$ then $f_1^2 = f_2^2 = f_3^2 = (1/4)t$, which does not satisfy any of the boundary conditions for symmetry SU(3) or Sp(2).

We may now take $\nu > 0$ and introduce the parameter change $r(t)^2 = f_1^2(t) f_3(t)^2$, with $r(t) > 0$. Then $|r'| = |(f_1 f_3)'| = |f_2|$ is strictly positive. Using (5.4) and (6.1), we get

$$f_1^2 = r\sqrt{\frac{r^2 - \mu^2}{r^2 + \nu^2 - \mu^2}}, \qquad f_2^2 = r^{-1}\sqrt{(r^2 - \mu^2)(r^2 + \nu^2 - \mu^2)},$$

$$f_3^3 = r\sqrt{\frac{r^2 + \nu^2 - \mu^2}{r^2 - \mu^2}}$$

and

$$\mathrm{d}t^2 = \frac{r\,\mathrm{d}r^2}{\sqrt{(r^2 - \mu^2)(r^2 + \nu^2 - \mu^2)}}.$$

These are ‘triaxial’ metrics with holonomy G_2 and SU(3)-symmetry. To be complete there must be a special orbit. This requires $f_1 = 0$ and $f_2, f_3 \neq 0$ at $t = 0$. The first condition implies $r(0)^2 = \mu^2$, the second gives $\mu = 0$. Thus this solution has $f_2^2(t) = f_3^2(t)$, which is the metric found by Bryant and Salamon [7] on the bundle of anti-self-dual two-forms over $\mathbb{C}P(2)$. This solution also defines a structure with Sp(2)-symmetry.

Theorem 9.3. *The space* $\mathbb{R} \times F_{1,2}$ *admits a one-parameter family of holonomy* G_2 *metrics with* SU(3)*-symmetry. Only one metric extends to a complete metric, and the underlying manifold is* $[\mathbb{C}P(2)_1|F_{1,2})$, *the bundle of anti-self-dual two-forms over* $\mathbb{C}P(2)$.

The space $\mathbb{R} \times \mathcal{F}_{(123)}$ *admits a unique incomplete metric with holonomy* G_2 *invariant under* SU(3).

The space $\mathbb{R} \times \mathbb{C}P(3)$ *admits two metrics with holonomy* G_2*. One is incomplete, the other extends to a complete metric on* $[S^4|\mathbb{C}P(3))$, *the bundle of anti-self-dual two-forms over* S^4.

Remark 9.4. As Andrew Dancer pointed out to us the substitutions $\mathrm{d}t = f_1 f_2 f_3\, \mathrm{d}s$ and $w_i = f_j f_k$ for each even permutation (ijk) of (123) reduce the SU(3)-symmetric holonomy G_2-equations to Euler's equations for a spinning top. These equations may then be solved by elliptic integrals. However, as this is no longer an arc-length parameterisation, one now has to work harder to determine questions of completeness.

Finally, we consider Eq. (5.5) for a symplectic G_2-structure with symmetry SU(3). We have $\sin\theta = 0$. Put $\varepsilon_\theta = \cos\theta$ and set

$$h^3 = f_1 f_2 f_3, \qquad x = f_2^{-2} h^2 \quad \text{and} \quad y = f_3^{-2} h^2,$$

so x and y are positive. Eq. (5.5) then give

$$6\varepsilon_\theta h' = xy + \frac{1}{x} + \frac{1}{y}. \tag{9.2}$$

On $(0,\infty)^2$, the right-hand side has a global minimum at $(1,1)$ and so $|h'| \geq 1/2$. This implies that there are no periodic solutions and that any complete solution has exactly one special orbit. As $f_2 f_3$ is even, we also see that f_1 vanishes at the special orbit. Therefore, we have exactly the same topologies as for holonomy G_2. Note, however, that there are more solutions to the symplectic equations than for holonomy G_2. A particularly simple example of this is furnished by setting $f_1(t) = t$, $f_2^2(t) = 1 + (t/2)^2$ and $f_2 \equiv f_3$. Complete triaxial solutions may be obtained as follows: begin with the complete U(3)-symmetric metric with holonomy G_2; hold h fixed, make a smooth deformation of x on $[1,\infty)$ and determine the corresponding deformation of y by (9.2).

Proposition 9.5. *Let* M^7 *admit a* G_2*-structure which is preserved by a cohomogeneity-one action of a compact simple Lie group. Then,* M^7 *admits an invariant symplectic* G_2*-structure if and only if* M^7 *admits an invariant metric with holonomy* G_2*. Similarly, complete symplectic* G_2*-structures only exist on manifolds with complete* G_2 *holonomy metrics.*

10. Smoothness of the three-form

In this section, we will briefly indicate how to check that the three-form ϕ is smooth once we have $h^*\phi_t = \phi_{-t}$ and smoothness of the metric g. The only case where significant work is required is that of special orbit $\mathbb{C}P(2)$ under SU(3)-symmetry. The case of special orbit S^4 under Sp(2)-symmetry follows by similar arguments.

The manifold $[\mathbb{C}P(2)|F_{1,2})$ is SU(3)-equivariantly isomorphic to the bundle of anti-self-dual two-forms Λ^2_- over $\mathbb{C}P(2)$. Bryant and Salamon [7] showed how to construct holonomy G_2 metrics on Λ^2_-, but they did not write down the general SU(3)-invariant three-form because they treated all four manifolds at once. In the following, we specialise Bryant and Salamon's approach in the style of Swann [21].

If we write $\mathbb{C}P(2) = \mathrm{SU}(3)/\mathrm{U}(2)$, then $P = \mathrm{SU}(3)$ is a principal bundle of frames with structure group $\mathrm{U}(2) = \mathrm{U}(1) \times_{\mathbb{Z}/2} \mathrm{Sp}(1)$. Under the action of U(2), we have $\Lambda^{1,0} \cong HL + \bar{L}^2$, where $L \cong \mathbb{C}$ and $H \cong \mathbb{C}^2$ are the standard representations of U(1) and Sp(1), respectively. This may be regarded as an identification not only of representations but also of bundles over $\mathbb{C}P(2)$, if to a representation V of U(2) we associate the bundle, also denoted V,

$$P \times_{\mathrm{U}(2)} V,$$

which is $P \times V$ modulo the action $(u, \xi) \mapsto (u \cdot g, g^{-1} \cdot \xi)$. We then have $\Lambda^2_- = S^2H \cong \mathrm{Im}\,\mathbb{H}$. Let $\boldsymbol{\theta} = \theta_0 + \theta_1 \mathrm{i} + \theta_2 \mathrm{j} + \theta_3 \mathrm{k} \in \Omega^1(P, \mathbb{H})$ be the canonical one-form. Write $\eta = \eta_1 \mathrm{i} + \eta_2 \mathrm{j} + \eta_3 \mathrm{k} \in \Omega^1(P, \mathrm{Im}\,\mathbb{H})$ for the $\mathfrak{sp}(1)$-part of the U(2) Levi-Civita connection. As the Fubini–Study metric is self-dual and Einstein, one finds that

$$\mathrm{d}\eta + \eta \wedge \eta = c\bar{\boldsymbol{\theta}} \wedge \boldsymbol{\theta}$$

for some positive constant c (a positive constant multiple of the scalar curvature). If $x = x_1 \mathrm{i} + x_2 \mathrm{j} + x_3 \mathrm{k}$ is the coordinate on $\mathrm{Im}\,\mathbb{H}$ then let $r^2 = x\bar{x} = -x^2$. The one-form

$$\boldsymbol{\alpha} = \mathrm{d}x + \eta x - x\eta$$

is semi-basic on $P \times S^2H$. One may now check that

$$\omega_1 = r^{-3}\boldsymbol{\alpha} x \boldsymbol{\alpha}, \qquad \omega_2 = 4c(r^{-1}\boldsymbol{\theta} x \bar{\boldsymbol{\theta}} + \bar{\boldsymbol{\theta}} \mathrm{i} \boldsymbol{\theta}), \qquad \omega_3 = 4c(r^{-1}\boldsymbol{\theta} x \bar{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}} \mathrm{i} \boldsymbol{\theta}),$$
$$\alpha = -4c(r^{-1}\boldsymbol{\theta}\boldsymbol{\alpha}\bar{\boldsymbol{\theta}} + r^{-3}\boldsymbol{\theta} x \boldsymbol{\alpha} x \bar{\boldsymbol{\theta}}), \qquad \beta = -4cr^{-2}\boldsymbol{\theta}(\boldsymbol{\alpha} x - x\boldsymbol{\alpha})\bar{\boldsymbol{\theta}}$$

satisfy Eq. (5.1) and hence define the required invariant forms on Λ^2_-.

To determine whether a particular form ϕ given by (5.3) is smooth on Λ^2_-, consider the pull-back of ϕ to $P \times S^2H$. There smoothness reduces to a question of smooth forms on $S^2H = \mathbb{R}^3$. Writing these forms in terms of $\mathrm{d}x_1$, $\mathrm{d}x_2$ and $\mathrm{d}x_3$ one now applies the results of Glaeser [12], see also [18], to determine the conditions for the coefficients to be smooth. Once g is smooth and $h^*\phi_t = \phi_{-t}$ one finds that there are no extra conditions.

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References

[1] L. Bérard Bergery, Sur de nouvelles variétés riemanniennes d'Einstein, Vol. 6, Publications de l'Institut É. Cartan, Nancy, 1982, pp. 1–60.
[2] A.L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Folge, Vol. 10, Springer, Berlin, 1987.
[3] A. Brandhuber, J. Gomis, S.S. Gubser, S. Gukov, Gauge theory at large N and new G_2 holonomy metrics, June 2001. eprint arXiv:hep-th/0106034.
[4] G.E. Bredon, Introduction to compact transformation groups, Pure Appl. Math. 46 (1972).
[5] R.L. Bryant, Metrics with exceptional holonomy, Ann. Math. 126 (1987) 525–576.
[6] R.L. Bryant, Classical, exceptional, and exotic holonomies: a status report, Actes de la Table Ronde de Géométrie Différentielle, Luminy, 1992, Paris, Sémin. Congr., Vol. 1, Soc. Math. France, 1996, pp. 93–165.
[7] R.L. Bryant, S.M. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989) 829–850.
[8] F.M. Cabrera, M.D. Monar, A.F. Swann, Classification of G_2-structures, J. London Math. Soc. 53 (1996) 407–416.
[9] R. Cleyton, G-structures and Einstein metrics, Ph.D. Thesis, University of Southern Denmark, Odense, 2001. ftp://ftp.imada.sdu.dk/pub/phd/2001/24.PS.gz.
[10] M. Cvetič, G.W. Gibbons, H. Lü, C.N. Pope, New cohomogeneity-one metrics with Spin(7) holonomy, May 2001. eprint arXiv:math.DG/0105119.
[11] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, On nearly parallel G_2-structures, J. Geom. Phys. 23 (3–4) (1997) 259–286.
[12] G. Glaeser, Fonctions composées différentiables, Ann. Math. 77 (2) (1963) 193–209.
[13] A. Gray, Weak holonomy groups, Math. Z. 123 (1971) 290–300.
[14] N.J. Hitchin, Stable forms and special metrics, in: M.A. Fernández, J.A. Wolf (Eds.), Global Differential Geometry: The Mathematical Legacy of Alfred Gray, Am. Math. Soc., Providence, RI, 2001.
[15] D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I, J. Diff. Geom. 43 (1996) 291–328.
[16] D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. II, J. Diff. Geom. 43 (1996) 329–375.
[17] D. Joyce, Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs, Oxford University Press, 2000.
[18] J.L. Kazdan, F.W. Warner, Curvature functions for open 2-manifolds, Ann. Math. 99 (2) (1974) 203–219.
[19] A. Kovalev, Twisted connected sums and special Riemannian holonomy, Preprint MS-00-011, University of Edinburgh, December 2000. eprint arXiv:math.DG/0012189.
[20] P.S. Mostert, On a compact Lie group acting on a manifold, Ann. Math. 65 (1957) 447–455;
P.S. Mostert, On a compact Lie group acting on a manifold, Ann. Math. 66 (1957) 589.
[21] A.F. Swann, Hyper-Kähler and quaternionic Kähler geometry, Math. Ann. 289 (1991) 421–450.